Math 3236 Statistical Theory

2/9/23

X_i random sample $i = 1 \ldots N$

X_i are i.i.d

indep. and ident. distributed

p.d.f. $f(x \mid \theta) = f(x;\theta)$

$= f_\theta(x)$

A function $r(\underline{X})$

a statistics

$r(\underline{X})$ is an estimator for θ (or a function of θ) if we expect $r(\underline{X})$ to be close to θ.

$\hat{\theta} = r(\underline{X})$ estimator

$r(\underline{x})$ estimate

$\hat{\theta}$ is consistent if

$$\hat{\theta}_N \xrightarrow{P} \theta \quad \text{as } N \to \infty$$

$\mathbb{E}(\hat{\theta}) = \theta$

$\mathrm{var}(\hat{\theta}) = \sigma^2_{\hat{\theta}}$

if $\sigma^2_{\hat{\theta}} \to 0$ as $N \to \infty$

The $\hat{\theta}$ is consistent.

$\mathbb{E}(\hat{\theta}) = \theta$

$\hat{\theta}(\underline{x}) = r(\underline{x})$

$$\int r(\underline{x})\, f(\underline{x};\theta)\, d\underline{x} = \theta \quad \forall \theta.$$

$$f(\underline{x};\theta) = \prod_{i=1}^{N} f(x_i;\theta)$$

Example.

X_i are uniform in $[0, A]$ with A unknown.

$$E(X_i) = \frac{A}{2}$$

$$\frac{1}{N} \sum_i X_i = \bar{X} = \frac{A}{2}$$

$$\hat{A} = 2\bar{X}$$

$$Var(X_i) = \frac{A^2}{12}$$

$$Var(\bar{X}) = \frac{A^2}{12N}$$

$$Var(\hat{A}) = \frac{A^2}{3N}$$

for N large

$$\hat{A} \simeq \mathcal{N}\left(A, \frac{A^2}{3N}\right)$$

$$\mathbb{P}\left(-z_{\alpha/2} \le \frac{\hat{A} - A}{A/\sqrt{3N}} \le z_{\alpha/2}\right) = 1-\alpha$$

$$\mathbb{P}(Z \le z_\alpha) = \alpha \qquad \text{if } Z \simeq \mathcal{N}(0,1)$$

$$\hat{A} \le A + z_{\alpha/2}\frac{A}{\sqrt{3N}} = A\left(1 + \frac{z_{\alpha/2}}{\sqrt{3N}}\right)$$

$$\mathbb{P}\left(\frac{\hat{A}}{\left(1 + z_{\alpha/2}/\sqrt{3N}\right)} \le A \le \frac{\hat{A}}{\left(1 - z_{\alpha/2}/\sqrt{3N}\right)}\right) = 1-\alpha$$

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$$\hat{A}_2 = \max X_i$$

$$P(\hat{A}_2 \le a) = P(X_1 \le a)^N = \left(\frac{a}{A}\right)^N$$

$$f_{\hat{A}_2}(a) = N \frac{a^{N-1}}{A^N}$$

$$E(\hat{A}_2) = \int_0^A a N \frac{a^{N-1}}{A^N} \, da = \frac{N}{A^N} \int_0^A a^N = \frac{A^{N+1}}{N+1} \frac{N}{A^N}$$

$$= \frac{N}{N+1} A$$

$$\hat{A}_3 = \left(1 + \frac{1}{N}\right) \hat{A}_2$$ unbiased.

$$\mathbb{E}(\hat{A}_2^2) = \int_0^A a^2 N \frac{a^{N-1}}{A^N} \, da =$$

$$= \frac{N}{A^N} \int a^{N+1} da =$$

$$= \frac{N}{A^N} \frac{A^{N+2}}{N+2} = \frac{N}{N+2} A^2$$

$$Var(\hat{A}_2) = A^2 \left(\frac{N}{N+2} - \frac{N^2}{(N+1)^2} \right)$$

$$= A^2 \frac{N(N+1)^2 - N^2(N+2)}{(N+2)(N+1)^2}$$

$$= \frac{N^3 + 2N^2 + N - N^2 - 2N^2}{(N+2)(N+1)^2} A^2$$

$$= \frac{N}{(N+2)(N+1)^2} A^2$$

$$\mathrm{var}\ \hat{A}_2 = \frac{N}{(N+2)(N+1)^2} A^2$$

$$\mathrm{var}\ \hat{A}_3 = \frac{1}{(N+2)N} A^2$$

$$\hat{A}_3 = \left(1 + \frac{1}{N}\right) \max_i X_i$$

$$f(\underline{x}; A) = \ell(A) \quad \text{likelihood function}$$

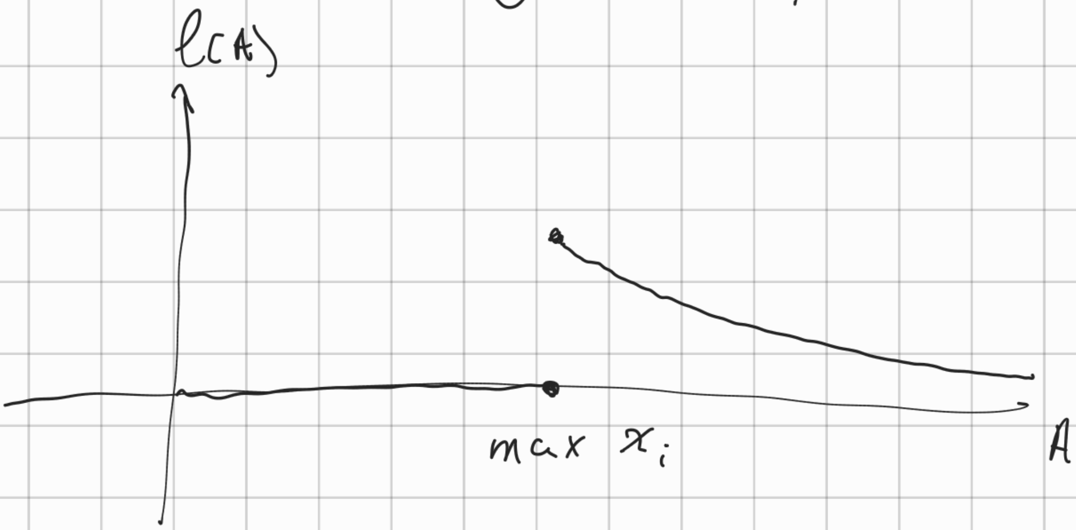
$$\hat{A}_{ML} = r(X)$$

$$f(\underline{x}, r(x)) = \max_A f(\underline{x}; A)$$

$$f(\underline{x}; A) = \prod_{i=1}^{N} f(x_i; A) =$$

$$= \begin{cases} \frac{1}{A^N} & \text{if all } x_i < A \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{1}{A^N} & A \geq \max x_i \\ 0 & \text{otherwise} \end{cases}$$



$$\hat{A}_{ML} = \max X_i$$

A $\quad l(A) = 0$ $\quad \Pi$